Def: A discrete valuation ring (DVR) is a ring of the form $\mathcal{O}_v$ for v some d.v. on some field.

Thm 8.2 For a ring A, TFAE:

(1) A is a DVR

(2) A is a local PID, but not a field.

(3) A is a UFD w. a unique prime element (up to associativity)

Proof: $(1) \Rightarrow (2)$ by Prop 8.1 $(\mathcal{O}_v \neq K$ since $\exists x \in K: v(x) < 0)$

$(2) \Rightarrow (3)$ $\mathrm{Spec}(A) = \{0 \neq P\}$ with $P = (\pi)$ for some prime elt. π.

Since A is a PID, it is a UFD. Prime elts. generate prime ideals, so π is the only one (up to associativity)

$(3) \Rightarrow (1)$ Each $0 \neq x \in \mathfrak{Q}(A)$ is of the form $x = \varepsilon \pi^k$ with $\varepsilon \in A^\times$, $k \in \mathbb{Z}$ $(k = v_\pi(a))$. Since $x \in A \Leftrightarrow k \geq 0$, $A = \mathcal{O}_{v_\pi}$. $\square$

Thm 8.3 For a noetherian local ring A, TFAE:

(1) A is a DVR

(2) A is a PID, but not a field

(3) A is a 1-dim UFD

(4) A is a 1-dim integrally closed domain

(5) A is a 1-dim regular ring.

Proof: $(1) \Rightarrow (2)$ T8.2 $(2) \Rightarrow (3)$ PIDs are UFDs [Q.3] and $\dim A \leq 1$. Since A is not a field, $\dim A = 1$.

$(3) \Rightarrow (4)$ UFDs are integrally closed [P6.4]

$(4) \Rightarrow (5)$ $\dim A = 1$, A local domain $\Rightarrow \mathrm{Spec}(A) = \{0 \neq M\}$.

Fix $a \in M \setminus \{0\} \Rightarrow \sqrt{(a)} = M \overset{C4.10}{\Rightarrow} \exists n \geq 0: M^n \subseteq (a)$

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Choose $n \geq 0$ <u>minimal</u> s.t. $M^n \subseteq (a)$. Since $a \notin A^\times$, $n \geq 1$.

$\Rightarrow M^{n-1} \nsubseteq (a)$. Choose $b \in M^{n-1} \setminus (a)$.

Define $\pi := \frac{a}{b} \in \mathcal{F}(A)$.

<u>Claim:</u> $\pi \in A$ and $M = (\pi)$.

$$\frac{1}{\pi} M = \frac{b}{a} M \subseteq \frac{1}{a} M^n \subseteq \frac{1}{a}(a) \subseteq A \Rightarrow \frac{1}{\pi} M \trianglelefteq A$$

Suppose $\underline{\frac{1}{\pi} M \neq A} \Rightarrow \frac{1}{\pi} M$ is contained in the (unique) max. ideal, i.e,

$\frac{1}{\pi} M \subseteq M$. A noeth. $\Rightarrow M$ f.g., faithful $\overset{P6.1(d)}{\Rightarrow} \frac{1}{\pi}$ integral$/A$

$\overset{A\text{ i.c.}}{\Longrightarrow} \frac{1}{\pi} \in A \Rightarrow b = \frac{1}{\pi} a \in (a)$ ↯ choice of b.

So $\underline{A = \frac{1}{\pi} M} \Rightarrow \boxed{M = \pi A}$

$\Rightarrow M/M^2 = \pi A / \pi^2 A \cong A/\pi A$ is 1-dimensional, so A is regular!

(5) $\Rightarrow$ (1) Let M be the max. ideal of A

A regular local of dim. $1 \overset{L7.13}{\Longrightarrow} M$ principal, say $M = (\pi)$, $\pi \in A \setminus \{0\}$.

<u>Note:</u> π is prime, because it generates a nonzero prime ideal.

<u>Claim:</u> $\bigcap_{k \geq 0} (\pi^k) = \underline{0}$ $(*)$

Suppose $0 \neq a \in \bigcap_{k \geq 0} (\pi^k)$. $\Rightarrow \forall k \geq 0: \ a\pi^{-k} \in A$

$\Rightarrow (a) \subsetneq (a\pi^{-1}) \subsetneq (a\pi^{-2}) \subsetneq \dots$ is an infinite asc. chain (proper bec. $\pi \notin A^\times$)

↯ A noetherian. $\square$(Claim).

Let $0 \neq a \in A$. Let $k \geq 0$ be maximal s.t. $a \in (\pi^k)$ ($\exists$ by $(*)$)

$\Rightarrow a = b\pi^k$ with $b \in A$.

$b \notin (\pi)$, by maximality of $k \overset{A\text{ local}}{\Longrightarrow} b \in A^\times$

<u>So:</u> A is a UFD w. unique prime π, hence $A = \mathcal{O}_{v_\pi}$. $\square$

<u>Remark</u> T8.3 shows for 1-dim noetherian domains A

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integrally closed $\Leftrightarrow$ regular (at each $P \in \mathrm{Spec}(A)$)

„$\Rightarrow$" fails in higher dimensions, but „$\Leftarrow$" holds, even

(Auslander-Buchsbaum Thm) Every regular local ring is a UFD